Practice problems (don't turn in):

1. [DPV] Problem 7.1 and:
   Can you use the dual LP to prove it's optimal?
2. [DPV] Problem 7.4 (LP for Duff beer)
3. [DPV] Problem 7.5 (LP for canine products)
4. [DPV] Problem 7.6: Give an example of an LP with unbounded feasible region but bounded optimum.
5. [DPV] Problem 7.11 (dual to the example)
6. [DPV] Problem 7.12 (prove that point (1.5,.5,0) is optimal.
7. (*Another reduction from 3SAT to a graph problem.*) Prove the following problem is `NP-hard`: Given an undirected graph $G = (V, E)$ and an integer $k > 0$, determine whether G contains a subset of k vertices whose induced subgraph is acyclic.

Problem 1 (MCQs on NP theory and LP.)

For each part, choose the answer that is *always* true. You do not need to explain your choice. Enter your responses directly on Gradescope.

Part (a): (2.5 points) Consider the problem of finding a 5−star graph:

Input: an undirected graph $G = (V, E)$.
output: a set of six vertices $\{u, v_1, v_2, \ldots, v_5\}$ such that $(uv_i) \in E$ for all $1 \leq i \leq 5$, and $(v_i v_j) \notin E$ for all $1 \leq, i, j \leq 5$, if such set exists, or return NO otherwise.

True or False: this problem is `NP-complete`.

A: True.
B: False.

Part (b): (2.5 points) Let $\mathcal{A}$, $\mathcal{B}$ be problems such that $\mathcal{A}$ is `NP-complete` and $\mathcal{B}$ does not belong to the class `NP`. You are told that $\mathcal{A} \rightarrow \mathcal{B}$. Which of the following are true? Circle all that apply.

A: $\mathcal{B}$ is `NP-hard`.
B: If an algorithm $\mathcal{L}$ efficiently solves $\mathcal{B}$, then there is an efficient algorithm to solve `SAT`.
C: Given an input $\mathcal{I}$ of $\mathcal{A}$, one can find a solution to it in polynomial time.

Part (c): (2.5 points) Consider an instance of LP with multiple solutions. This is, there are multiple vectors x *in the feasible region* such that $f(x) = c^T x$ is maximum. Let x_1, x_2 be two such solutions.

$\lambda x_1 + (1 - \lambda) x_2$ is also a solution for

A: All $\lambda \in \mathbb{R}$.
B: All $0 \leq \lambda \leq 1$.
C: All $\lambda > 0$.
D: None of the other options is correct.

Part (d): (3 points) Consider the following instance of LP:

$$\begin{aligned} \max \quad & x + y \\ s.t.: \quad & ax + by \leq 1 \\ & x, y \geq 0. \end{aligned}$$

where $a, b \in \mathbb{R}$ are parameters. This problem is infeasible if

A: $a > 0,\ b < 0$.
B: This instance is always feasible.
C: $a < 0, b < 0$.
D: $a > 0,\ b > 0$.

Part (e): (3 points) Consider the following instance of LP:

$$\begin{aligned} \max\ & x + y \\ s.t.:\ & ax + by \leq 1 \\ & x, y \geq 0. \end{aligned}$$

where $a, b \in \mathbb{R}$ are parameters. This problem is unbounded if

A: $a > 0,\ b < 0.$
B: This instance is always bounded.
C: $a < 0, b < 0.$
D: $a > 0,\ b > 0.$

Part (f): (3 points) Consider the following instance of LP:

$$\begin{aligned} \max\ & x + y \\ s.t.:\ & ax + by \leq 1 \\ & x, y \geq 0. \end{aligned}$$

where $a, b \in \mathbb{R}$ are parameters. This problem has a unique optimal solution (x, y) if

A: $a > 0,\ b > 0.$
B: $a = b$ and $a > 0.$
C: $a > 0,\ b > 0$ and $a \neq b.$
D: There is always a unique optimal solution.

Part (g): (3.5 points) Consider the following instance of LP:

$$\begin{aligned} & \max 5x_1 + 9x_2 \\ s.t.:\ & x_1 + x_2 \leq 3 \\ & x_1 + 3x + 2 \leq 7 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Let (x'_1, x'_2) be a point in the feasible region, and let (y'_1, y'_2) be a point in the feasible region of the Dual LP. Which of the following inequalities *must* be true? Check ALL that apply. .

A: $5x'_1 + 12x'_2 \leq 3y'_1 + 7y'_2$
B: $5x'_1 + 9x'_2 \leq 3y'_1 + 7y'_2$
C: $4x'_1 + 8x'_2 \leq 4y'_1 + 8y'_2$
D: $y'_1 + y'_2 \geq 5$

Problem 2 (NP-reduction)

[DPV 8.9] (Hitting Set). In the `Hitting Set` problem, we are given a family of sets $\{S_1, S_2, \ldots, S_n\}$ and a budget $b > 0$, and we wish to find a set H of size less or equal than b which intersects every S_i, if such an H exists.

Show that `Hitting Set` is `NP-complete`.